International Journal of Theoretical Physics, Vol. 45, No. 6, June 2006 (© 2006)
DOI: 10.1007/s10773-006-9109-0

Orthomodularity of Decompositions in a Categorical Setting

John Harding[1]

Received September 15, 2005; accepted March 5, 2006
Published Online: May 17, 2006

We provide a method to construct a type of orthomodular structure known as an orthoalgebra from the direct product decompositions of an object in a category that has finite products and whose ternary product diagrams give rise to certain pushouts. This generalizes a method to construct an orthomodular poset from the direct product decompositions of familiar mathematical structures such as non-empty sets, groups, and topological spaces, as well as a method to construct an orthomodular poset from the complementary pairs of elements of a bounded modular lattice.

KEY WORDS: orthomodular poset; orthoalgebra; decomposition; product; category.

Mathematics Subject Classifications (2000): 06C15, 81P10, 03G12, 18A30

1. INTRODUCTION

Since the work of Birkhoff and von Neumann (1936), various types of orthomodular structures generalizing the lattice $\mathcal{C}(\mathcal{H})$ of closed subspaces of a Hilbert space have been used as models for the propositions of a quantum mechanical system. This forms the basis of the quantum logic approach to quantum mechanics.

Among many ways to view the closed subspaces of a Hilbert space $\mathcal{H}$, one notes that they correspond exactly to direct product decompositions $\mathcal{H} \simeq \mathcal{H}_1 \times \mathcal{H}_2$ of the Hilbert space. Thus, the direct product decompositions of a Hilbert space $\mathcal{H}$ form an orthomodular structure that is isomorphic to $\mathcal{C}(\mathcal{H})$. One might expect that the source of orthomodularity in this construction is closely tied to properties of the Hilbert space. Harding (1996) has shown that this is not the case, and that the direct product decompositions of most familiar mathematical structures, such as non-empty sets, groups, and topological spaces, naturally form a type of orthomodular structure known as an orthomodular poset. In Harding (1999), Harding takes this result as a basic building block for an axiomatization of a fragment of quantum mechanics.

[1] Department of Mathematical Sciences, New Mexico State University, Las Cruces, NM 88003, USA; e-mail: jharding@nmsu.edu.

In this note we generalize Harding's result to show that the direct product decompositions of any object in a suitable type of category form a type of orthomodular structure known as an orthoalgebra. This categorical approach has a number of advantages. It provides a relatively simple method to determine when the decompositions of a given type of mathematical structure will yield an orthoalgebra; it provides a result with much wider applicability than the original; and this categorical construction unifies the earlier result on constructing orthomodular posets from the decompositions of various types of mathematical structures with a related result, described by Harding (1996) and Mushtari (1989), on constructing an orthomodular poset from the complementary pairs of elements of a bounded modular lattice.

There is one other advantage to this categorical approach to decompositions. There has been recent interest in using categorical methods to address foundational issues in quantum mechanics (Abramsky and Coecke, 2004; Isham and Butterfield, 2000). While I know of no direct links between the methods described here and the approaches of these authors, the results presented here do provide a link between quantum logic and a categorical viewpoint, and this may some day serve as a useful bridge.

This paper is organized in the following manner. In the second section we review some basics, and provide the definition of the type of category we will consider, the so-called honest categories. In the third section we describe how to construct an algebraic structure $\mathcal{D}(A)$ from the decompositions of an object A in an honest category and prove our main result, that $\mathcal{D}(A)$ is an orthoalgebra. In the fourth, and final, section, we provide an example to show that the structure $\mathcal{D}(A)$ need not be an orthomodular poset, and discuss possible directions of further study.

2. HONEST CATEGORIES

A finite product diagram in a category $\mathcal{C}$ is a finite sequence of morphisms $(f_1, \ldots, f_n)$ with $f_i : A \to A_i$ such that for each sequence $(g_1, \ldots, g_n)$ with $g_i : B \to A_i$ there is a unique $h : B \to A$ with $f_i \circ h = g_i$ for each $i = 1, \ldots, n$. A category $\mathcal{C}$ is said to have finite products if for each sequence $A_1, \ldots, A_n$ of objects, there is an object A and a product diagram $(f_1, \ldots, f_n)$ with $f_i : A \to A_i$.

Throughout, we shall work with a category $\mathcal{C}$ that has finite products. We shall further assume that for each finite sequence $A_1, \ldots, A_n$ of objects that we have selected a specific object, denoted $A_1 \times \cdots \times A_n$, and a specific product diagram, denoted $(\pi_1^{A_1 \times \cdots \times A_n}, \ldots, \pi_n^{A_1 \times \cdots \times A_n})$ with $\pi_i^{A_1 \times \cdots \times A_n} : A_1 \times \cdots \times A_n \to A_i$. Usually the domain of the maps $\pi_i^{A_1 \times \cdots \times A_n}$ is clear from the context, and we simply write $\pi_i : A_1 \times \cdots \times A_n \to A_i$ and call these the projection maps. Using these projections, for any sequence of morphisms $(f_1, \ldots, f_n)$ with $f_i : A \to A_i$, we define the morphism $f_1 \times \cdots \times f_n : A \to A_1 \times \cdots \times A_n$ to be the unique morphism

$$
\begin{tikzcd}
 & A \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\
A_1 \arrow[dr, "\tau_{A_1}"'] & & A_2 \arrow[dl, "\tau_{A_2}"] \\
 & \Omega &
\end{tikzcd}
$$

Fig. 1.

with $\pi_i \circ (f_1 \times \cdots \times f_n) = f_i$ for each $i = 1, \ldots, n$. We use $\pi_{ij}^{A_1\times\cdots\times A_n}$ as an abbreviation for $\pi_i^{A_1\times\cdots\times A_n} \times \pi_j^{A_1\times\cdots\times A_n}$.

We require the category $\mathcal{C}$ to have finite products. By this we mean that the empty family must have a product as well. In other words, $\mathcal{C}$ must have a terminal object. We assume that a specific terminal object Ω has been selected, and for each object A we let $\tau_A : A \to \Omega$ be the unique morphism to this terminal object.

We recall that a sequence (f_1, f_2, g_1, g_2) with $f_i : A \to A_i$, $g_i : A_i \to B$ and $g_1 \circ f_i = g_2 \circ f_2$ is called a pushout if for every pair of morphisms (u_1, u_2) with $u_i : A_i \to C$ and $u_1 \circ f_1 = u_2 \circ f_2$ there is a unique $h : B \to C$ with $h \circ g_i = u_i$. The relationship between pushouts and products is key to our definition of an honest category.

Definition 2.1. A product diagram $(f_1, \ldots, f_n)$ is called a disjoint product diagram if for each $i = 1, \ldots, n$ the sequence $(f_i, \prod_{j\neq i} f_j, \tau_{A_i}, \tau_{\prod_{j\neq i} A_j})$ is a pushout.

In particular, a binary product diagram (f_1, f_2) is disjoint if Fig. 1 is a pushout, and the ternary product diagram (f_1, f_2, f_3) is disjoint if Fig. 2, as well as two others formed by permuting indices, is a pushout.

$$
\begin{tikzcd}
 & A \arrow[dl, "f_1"'] \arrow[dr, "f_2 \times f_3"] & \\
A_1 \arrow[dr, "\tau_{A_1}"'] & & A_2 \times A_3 \arrow[dl, "\tau_{A_2\times A_3}"] \\
 & \Omega &
\end{tikzcd}
$$

Fig. 2.

$$\begin{tikzcd}
& A \arrow[dl, "f_1 \times f_3"'] \arrow[dr, "f_2 \times f_3"] & \\
A_1 \times A_3 \arrow[dr, "\pi_2"'] & & A_2 \times A_3 \arrow[dl, "\pi_2"] \\
& A_3 &
\end{tikzcd}$$

Fig. 3.

Definition 2.2. A category $\mathcal{C}$ is called an honest category if it has finite products, all projections are epic, and for each disjoint product diagram (f_1, f_2, f_3) the diagram $(f_1 \times f_3, f_2 \times f_3, \pi_2, \pi_2)$ is a pushout. See Fig. 3.

There are many examples of honest categories whose objects are based on sets and whose products are based on usual Cartesian products of sets; these include the category of non-empty sets, any category of algebras with finitary or infinitary operations, the category of topological spaces, the category of uniform spaces, and the category of posets. Other examples of honest categories include the category of sets with morphisms being relations, and the objects with global support in a topos. Any join semi-lattice with zero that satisfies the implication $a \wedge (b \vee c) = b \wedge (a \vee c) = c \wedge (a \vee b) = 0 \Rightarrow (a \vee c) \wedge (b \vee c) = c$, where the symbol $\wedge$ is meant to imply an existing meet, provides an honest category where products are given by joins and pushouts by meets. Any distributive lattice with zero is easily seen to provide an example of such a join-semilattice, and with modest effort one sees that any modular lattice with zero provides such a join-semilattice.

3. DECOMPOSITIONS IN AN HONEST CATEGORY

Let A be an object in an honest category $\mathcal{C}$. Define an equivalence relation $\simeq$ on the collection of all morphisms with domain A by setting $f \simeq g$ if there is an isomorphism u with $u \circ f = g$. Consider the set of all finite product diagrams $(f_1, \ldots, f_n)$ where the maps $f_i, \ldots, f_n$ have common domain A and define an equivalence relation $\approx$ on this collection of product diagrams by setting $(f_1, \ldots, f_n) \approx (g_1, \ldots, g_n)$ if $f_i \simeq g_i$ for each $i = 1, \ldots, n$. We use $[\, f_1, \ldots, f_n\,]$ to denote the equivalence class of $\approx$ containing $(f_1, \ldots, f_n)$ and call this equivalence class an n-ary decomposition of A. In the case that the product diagram $(f_1, \ldots, f_n)$ is disjoint, we call $[\, f_1, \ldots, f_n\,]$ a disjoint decomposition of A.

Lemma 3.1. *Suppose* $[h_1, h_2, h_3]$ *and* $[k_1, k_2, k_3]$ *are disjoint decompositions of* A *with* $[h_1, h_2 \times h_3] = [k_1, k_2 \times k_3]$ *and* $[h_2, h_1 \times h_3] = [k_2, k_1 \times k_3]$*, then* $[h_1, h_2, h_3] = [k_1, k_2, k_3]$.

Proof: Suppose $h_i : A \to A_i$ and $k_i : A \to B_i$. As $[h_1, h_2 \times h_3] = [k_1, k_2 \times k_3]$ there are isomorphisms $u_1 : A_1 \to B_1$ and $v : A_2 \times A_3 \to B_2 \times B_3$ with $u_1 \circ h_1 = k_1$ and $v \circ (h_2 \times h_3) = k_2 \times k_3$, and as $[h_2, h_1 \times h_3] = [k_2, k_1 \times k_3]$ there are isomorphisms $u_2 : A_2 \to B_2$ and $w : A_1 \times A_3 \to B_1 \times B_3$ with $u_2 \circ h_2 = k_2$ and $w \circ (h_1 \times h_3) = k_1 \times k_3$.

As (k_1, k_2, k_3) is a disjoint product diagram and the category $\mathcal{C}$ is honest, we have $(k_1 \times k_3, k_2 \times k_3, \pi_2, \pi_2)$ is a pushout diagram. Then as v, w are isomorphisms, $(w^{-1} \circ (k_1 \times k_3), v^{-1} \circ (k_2 \times k_3), \pi_2 \circ w, \pi_2 \circ v)$ is a pushout, and upon simplifying, $(h_1 \times h_3, h_2 \times h_3, \pi_2 \circ w, \pi_2 \circ v)$ is a pushout. But (h_1, h_2, h_3) is also a disjoint product diagram, so honesty gives $(h_1 \times h_3, h_2 \times h_3, \pi_2, \pi_2)$ is a pushout. So there is an isomorphism $u_3 : A_3 \to B_3$ with $u_3 \circ \pi_2 = \pi_2 \circ w$, hence $u_3 \circ \pi_2 \circ (h_1 \times h_3) = \pi_2 \circ w \circ (h_1 \times h_3)$, which gives $u_3 \circ h_3 = k_3$. Therefore there are isomorphisms u_1, u_2, u_3 with $u_i \circ h_i = k_i$ for $i = 1, 2, 3$, hence $[h_1, h_2, h_3] = [k_1, k_2, k_3]$. □

Definition 3.2. For A an object in an honest category $\mathcal{C}$, let $\mathcal{D}(A)$ be the set of disjoint binary decompositions $[f_1, f_2]$ of A. Define constants 0, 1 on $\mathcal{D}(A)$ by setting

$$0 = [\tau_A, 1_A] \quad \text{and} \quad 1 = [1_A, \tau_A].$$

Define a relation $\perp$ on $\mathcal{D}(A)$ and a partial binary operation $\oplus$ with domain $\perp$ as follows. Set $[f_1, f_2] \perp [g_1, g_2]$ if there is a disjoint ternary decomposition $[h_1, h_2, h_3]$ with

$$[h_1, h_2 \times h_3] = [f_1, f_2] \quad \text{and} \quad [h_2, h_1 \times h_3] = [g_1, g_2]$$

and in this case define

$$[f_1, f_2] \oplus [g_1, g_2] = [h_1 \times h_2, h_3].$$

The crucial definitions of $\perp$ and $\oplus$ express that the sum of two decompositions is defined when they are built in a certain way from a common refinement, and that their sum is constructed from this refinement. That $\oplus$ is well-defined follows from Lemma 3.1.

Definition 3.3. An orthoalgebra is a set X with constants 0 and 1, and a partial binary operation $\oplus$ which satisfies the following:

1. If $f \oplus g$ is defined, then $g \oplus f$ is defined and $f \oplus g = g \oplus f$.

2. If $f \oplus g$ is defined and $e \oplus (f \oplus g)$ is defined, then $e \oplus f$ is defined, $(e \oplus f) \oplus g$ is defined and $e \oplus (f \oplus g) = (e \oplus f) \oplus g$.
3. For each f in X, there is exactly one f^* in X with $f \oplus f^*$ defined and $f \oplus f^* = 1$.
4. If $f \oplus f$ is defined, then $f = 0$.

Orthoalgebras were introduced by Randall and Foulis in 1979 as a generalization of orthomodular posets that admits a tensor product. Since their inception, these structures have received a good amount of attention. For a detailed account of orthoalgebras the reader should consult (Foulis et al., 1992; Wilce, 2000), and for general background on orthomodular posets and lattices the reader should consult (Kalmbach, 1983; Ptak and Pulmannová, 1991).

Theorem 3.4. *For any object A object in an honest category $\mathcal{C}$, the structure* $(\mathcal{D}(A), \perp, \oplus, 0, 1)$ *is an orthoalgebra.*

Proof: To verify the first condition in the definition of an orthoalgebra suppose $[f_1, f_2] \oplus [g_1, g_2]$ is defined. Then there is a disjoint decomposition $[h_1, h_2, h_3]$ with $[h_1, h_2 \times h_3] = [f_1, f_2]$ and $[h_2, h_1 \times h_3] = [g_1, g_2]$. Then $[h_2, h_1, h_3]$ is a disjoint decomposition showing $[g_1, g_2] \oplus [f_1, f_2]$ is defined, and as there is an isomorphism i with $i \circ (h_1 \times h_2) = h_2 \times h_1$ it follows that $[f_1, f_2] \oplus [g_1, g_2] = [g_1, g_2] \oplus [f_1, f_2]$.

The second condition in the definition of an orthoalgebra requires more effort, and we return to it momentarily. For the third condition, we note that for a disjoint decomposition $[f_1, f_2]$, that $[f_2, f_1]$ is also a disjoint decomposition, and that $[f_1, f_2, \tau_A]$ is a disjoint decomposition showing $[f_1, f_2] \oplus [f_2, f_1]$ is defined and equal to $[1_A, \tau_A] = 1$. Suppose $[g_1, g_2]$ is a disjoint decomposition with $[f_1, f_2] \oplus [g_1, g_2]$ defined and equal to $[1_A, \tau_A]$. Then there is a disjoint decomposition $[h_1, h_2, h_3]$ with $[h_1, h_2 \times h_3] = [f_1, f_2]$, $[h_2, h_1 \times h_3] = [g_1, g_2]$ and $[h_1 \times h_2, h_3] = [1_A, \tau_A]$. It follows that $h_3 = \tau_A$. So there is an isomorphism i with $i \circ (h_2 \times h_3) = h_2$, and it follows that $[h_1, h_2] = [h_1, h_2 \times h_3] = [f_1, f_2]$. Thus $[h_1, h_2, h_3] = [f_1, f_2, \tau_A]$, and it follows that $[g_1, g_2] = [f_2, f_1]$.

For the fourth condition, suppose $[f_1, f_2] \oplus [f_1, f_2]$ is defined. Then there is a disjoint decomposition $[h_1, h_2, h_3]$ with $[h_1, h_2 \times h_3] = [f_1, f_2]$ and $[h_2, h_1 \times h_3] = [f_1, f_2]$. So $h_1 \simeq h_2$, and therefore $[h_1, h_1, h_3]$ is a disjoint decomposition. Suppose $h_i : A \to A_i$. Then disjointness yields $(h_1, h_1 \times h_3, \tau_{A_1}, \tau_{A_1 \times A_3})$ is a pushout. As $1_{A_1} \circ h_1 = h_1 = \pi_1 \circ (h_1 \times h_3)$ there is a map $i : \Omega \to A_1$ with $i \circ \tau_{A_1} = 1_{A_1}$. Since $\tau_{A_1} \circ i : \Omega \to \Omega$, as Ω is terminal we have $\tau_{A_1} \circ i = 1_\Omega$. So τ_{A_1} and i are mutually inverse isomorphisms. In particular $A_1 \simeq \Omega$, so A_1 is terminal. Thus $[h_1, h_1, h_3] = [\tau_A, \tau_A, h_3]$. As $[\tau_A, \tau_A, h_3]$ is a product diagram, h_3 is an isomorphism, and therefore $[f_1, f_2] = [\tau_A, 1_A] = 0$.

We now return to the second condition. Let $[e, e']$, $[f, f']$ and $[g, g']$ be disjoint decompositions (we shift notation slightly to ease readability) with $[f, f'] \oplus [g, g']$ defined, and $[e, e'] \oplus ([f, f'] \oplus [g, g'])$ is defined. Then there are disjoint decompositions $[h_1, h_2, h_3]$ and $[k_1, k_2, k_3]$ with

$$
\begin{aligned}
[h_1, h_2 \times h_3] &= [f, f'] \\
[h_2, h_1 \times h_3] &= [g, g'] \\
[h_1 \times h_2, h_3] &= [f, f'] \oplus [g, g']
\end{aligned}
$$

$$
\begin{aligned}
[k_1, k_2 \times k_3] &= [e, e'] \\
[k_2, k_1 \times k_3] &= [f, f'] \oplus [g, g'] \\
[k_1 \times k_2, k_3] &= [e, e'] \oplus ([f, f'] \oplus [g, g'])
\end{aligned}
$$

The above data, with the observation that $[f, f'] \oplus [g, g']$ is equal to both $[h_1 \times h_2, h_3]$ and $[k_2, k_1 \times k_3]$, gives

$$h_1 \simeq f, h_2 \simeq g, k_1 \simeq e, h_3 \simeq k_1 \times k_3 \simeq e \times k_3 \quad \text{and} \quad k_2 \simeq h_1 \times h_2 \simeq f \times g.$$

Then as $[h_1, h_2, h_3]$ and $[k_1, k_2, k_3]$ are disjoint decompositions we have

$$[f, g, e \times k_3] \text{ and } [e, f \times g, k_3] \text{ are disjoint decompositions.}$$

Then from general properties of products, $[e, f, g, k_3]$ is a decomposition, and therefore $[e, f, g \times k_3]$ is a decomposition. We next show that $[e, f \times g, k_3]$ is disjoint. It will be convenient to suppose $e : A \to E$, $f : A \to F$, $g : A \to G$ and $k_3 : A \to K_3$. □

Claim $[e, f, g \times k_3]$ is a disjoint decomposition.

Proof: Note that $[e, f \times g, k_3]$ being disjoint implies $[e, f \times g \times k_3]$ is disjoint. Also $[f, g, e \times k_3]$ being disjoint implies $[f, e \times g \times k_3]$ is disjoint. It remains to show that $[e \times f, g \times k_3]$ is disjoint, that is, that $(e \times f, g \times k_3, \tau, \tau)$ is a pushout.

Suppose u, v are as shown in Fig. 4 with $u \circ (e \times f) = v \circ (g \times k_3)$. We must show there is a unique $\lambda : \Omega \to D$ completing this diagram. As $u \circ (e \times f) = v \circ (g \times k_3)$ we have $u \circ \pi_{12} \circ (e \times f \times k_3) = v \circ \pi_{23} \circ (e \times g \times k_3)$, so the outside square in Fig. 5 commutes, and similarly the outside square in Fig. 6 commutes.

As $[f, g, e \times k_3]$ and $[e, f \times g, k_3]$ are disjoint decompositions, honesty gives that the inside squares in Fig. 5 and 6 are a pushouts. Therefore there are unique maps φ and ψ completing these diagrams in the indicated manner.

Using the fact that $[e, f \times g, k_3]$ is disjoint, from the definition of disjointness we have that the inside square in Fig. 7 is a pushout, and a bit of diagram chasing involving Figs. 5 and 6 shows that the outside square of Fig. 7 is commutative. Therefore there is a unique map $\lambda : \Omega \to D$ completing Fig. 7.

$$
\begin{tikzcd}
& A \arrow[dl, "e \times f"'] \arrow[dr, "g \times k_3"] & \\
E \times F \arrow[dr, "\tau"] \arrow[ddr, "u"'] & & G \times K_3 \arrow[dl, "\tau"'] \arrow[ddl, "v"] \\
& \Omega \arrow[d, dotted, "\lambda"'] & \\
& D &
\end{tikzcd}
$$

Fig. 4.

As Ω is terminal, there is a unique map from $E \times F \times K_3$ to Ω. Therefore we have $\lambda \circ \tau_{E\times F} \circ \pi_{12} = \lambda \circ \tau_{E\times K_3} \circ \pi_{13}$. Using Fig. 7, this composition is equal to $\varphi \circ \pi_{13}$, which by Fig. 5 equals $u \circ \pi_{12}$. Part of the definition of honesty requires all projections be epimorphisms, so $\lambda \circ \tau_{E\times F} \circ \pi_{12} = u \circ \pi_{12}$ yields $\lambda \circ \tau_{E\times F} = u$. Similarly $\lambda \circ \tau_{G\times K_3} = v$, showing λ does indeed complete Fig. 4.

It remains to show λ is the unique map completing Fig. 4. Suppose λ' is another. Then as there is a unique map from $E \times F \times K_3$ to the terminal object Ω, we have $\lambda' \circ \tau_{E\times K_3} \circ \pi_{13} = \lambda' \circ \tau_{E\times F} \circ \pi_{12}$. By our assumption on λ', this composition is equal to $u \circ \pi_{12}$, and by Fig. 5 this equals $\varphi \circ \pi_{13}$. As projections are epic and $\lambda' \circ \tau_{E\times K_3} \circ \pi_{13} = \varphi \circ \pi_{13}$ we have $\lambda' \circ \tau_{E\times K_3} = \varphi$, and similarly $\lambda' \circ \tau_{F\times G} = \psi$. Then from the uniqueness of the map completing Fig. 7, we have $\lambda = \lambda'$. This concludes the proof of the claim.

We have shown that $[\, e, f, g \times k_3 \,]$ is a disjoint decomposition. As we have seen that $f \times g \simeq k_2$, we have $f \times g \times k_3 \simeq k_2 \times k_3 \simeq e'$. Also, as $e \times k_3 \simeq h_3$

$$
\begin{tikzcd}
& A \arrow[dl, "e \times f \times k_3"'] \arrow[dr, "e \times g \times k_3"] & \\
E \times F \times K_3 \arrow[dr] \arrow[ddr, "u \circ \pi_{12}"'] & & E \times G \times K_3 \arrow[dl] \arrow[ddl, "v \circ \pi_{23}"] \\
& E \times K_3 \arrow[d, dotted, "\varphi"'] & \\
& D &
\end{tikzcd}
$$

Fig. 5.

$$
\begin{tikzcd}
& A \arrow[dl, "e\times f\times g"'] \arrow[dr, "f\times g\times k_3"] & \\
E\times F\times G \arrow[dr] \arrow[ddr, "u\circ\pi_{12}"'] & & F\times G\times K_3 \arrow[dl] \arrow[ddl, "v\circ\pi_{23}"] \\
& F\times G \arrow[d, dotted, "\psi"'] & \\
& D &
\end{tikzcd}
$$

Fig. 6.

and $g \simeq h_2$ we have $e \times g \times k_3 \simeq h_2 \times h_3 \simeq f'$. Thus

$$
\begin{aligned}
[\, e, f \times g \times k_3\,] &= [\, e, e'\,] \\
[\, f, e \times g \times k_3\,] &= [\, f, f'\,]
\end{aligned}
$$

This shows $[\, e, e'\,] \oplus [\, f, f'\,]$ is defined and equal to $[\, e \times f, g \times k_3\,]$.

We next see that $[\, e \times f, g, k_3\,]$ is disjoint. Earlier we noted that $[\, e, f \times g, k_3\,]$ and $[\, f, g, e \times k_3\,]$ are disjoint. From these we obtain that $[\, e \times f \times g, k_3\,]$ and $[\, g, e \times f \times k_3\,]$ are disjoint. We have just shown that $[\, e, f, g \times k_3\,]$ is disjoint, and this yields that $[\, e \times f, g \times k_3\,]$ is disjoint. This shows that $[\, e \times f, g, k_3\,]$ is disjoint.

Above we have noted that $f \simeq h_1$ and $e \times k_3 \simeq h_3$. So $e \times f \times k_3 \simeq h_1 \times h_3 \simeq g'$. This, and the above description of $[\, e, e'\,] \oplus [\, f, f'\,]$ gives

$$
\begin{aligned}
[\, e, e'\,] \oplus [\, f, f'\,] &= [\, e \times f, g \times k_3\,] \\
[g, g'] &= [\, g, e \times f \times k_3\,]
\end{aligned}
$$

This shows that $([\, e, e'\,] \oplus [\, f, f'\,]) \oplus [\, g, g'\,]$ is defined and equal to $[\, e \times f \times g, k_3\,]$. As $e \simeq k_1$ and $f \times g \simeq k_2$ we have $e \times f \times g \simeq k_1 \times k_2$. Therefore, as

$$
\begin{tikzcd}
& A \arrow[dl, "e\times k_3"'] \arrow[dr, "f\times g"] & \\
E\times K_3 \arrow[dr] \arrow[ddr, "\varphi"'] & & F\times G \arrow[dl] \arrow[ddl, "\psi"] \\
& \Omega \arrow[d, dotted, "\lambda"'] & \\
& D &
\end{tikzcd}
$$

Fig. 7.

$[e, e'] \oplus ([f, f'] \oplus [g, g'])$ is given above by $[k_1 \times k_2, k_3]$ we have that the two expressions are equal. □

4. FURTHER REMARKS

Harding (1996) has shown that the direct product decompositions of any nonempty set, group, topological space, etc. form a type of orthomodular structure known as an orthomodular poset. In this same paper Harding exhibits several related constructions, namely, a method to produce an orthomodular poset from the equivalence elements of any relation algebra, and one to produce an orthomodular poset from the complementary pairs of elements of any bounded modular lattice. Mushtari 1989 has shown that one can construct an orthomodular poset from the complementary pairs of any bounded lattice that is both M-symmetric and M*-symmetric.

The above categorical viewpoint encompasses several of these methods. The categories of non-empty sets, groups, and topological spaces are all honest (with all products disjoint), and the construction described above reduces to that described by Harding in these cases. Also, any modular lattice forms an honest category, and the construction described above reduces to that described by Harding and Mushtari in this case as well.

All this raises several questions. First, one might wonder whether the above construction yields not just an orthoalgebra, but the more specialized type of structure known as an orthomodular poset. An orthomodular poset, in our current situation, is most easily described as an orthoalgebra in which $e \oplus f$, $e \oplus g$, and $f \oplus g$ being defined implies $(e \oplus f) \oplus g$ is defined (the standard treatment of orthomodular posets (Kalmbach, 1983; Pták and Pulmannová, 1991) defines them as a special type of orthocomplemented poset). The example below shows that the above construction does not in general yield orthomodular posets.

Proposition 4.1. *There is a lattice L, that when considered as a category, is honest and contains an object A with $\mathcal{D}(A)$ an orthoalgebra that is not an orthomodular poset.*

Proof: We view a lattice as a category where there is a unique morphism from x to y when $x \geq y$. The join of two objects x, y is their product, and their meet is their pushout. Our lattice will have a lower bound 0 which is the terminal object, and an upper bound 1 which will serve as our object A. For a lattice to be honest, when viewed as a category, it must satisfy for all x, y, z if $x \wedge (y \vee z) = y \wedge (x \vee z) = z \wedge (x \vee y) = 0$, then $(x \vee z) \wedge (y \vee z) = z$.

The lattice L we consider will be a sixteen element Boolean algebra with one coatom removed. To be specific, suppose B is a sixteen element Boolean algebra with atoms a, b, c, d and let $L = B - \{d'\}$ with the partial ordering inherited from B. Note that L is a subset of B, but not a sublattice, that all meets in L agree with those in B, and that all joins in L that do not evaluate to 1 agree with those in B. As B is Boolean it is clearly honest. Suppose x, y, z belong to L and satisfy $x \wedge (y \vee z) = y \wedge (x \vee z) = z \wedge (x \vee y) = 0$. If none of $x \vee y$, $x \vee z$ or $y \vee z$ evaluate to 1, then these joins all agree with those in 2^4, and it follows from the honesty of 2^4 that $(x \vee z) \wedge (y \vee z) = z$, and if any of these joins does evaluate to 1, then one fairly trivially obtains $(x \vee z) \wedge (y \vee z) = z$. So L is honest.

We next describe elements e, f, g of $\mathcal{D}(A)$. Each element will be a disjoint binary decomposition of 1. Rather than stating the two morphisms (with domain 1) that describe this binary decomposition, we specify only the codomains of these morphisms, as our category is a lattice. Thus each of e, f, g will be an ordered pair of complementary elements of L. We set $e = (a, a')$, $f = (b, b')$ and $g = (c, c')$. Then $(a, b, a' \wedge b')$ is a disjoint decomposition showing that $e \oplus f$ is defined, $(a, c, a' \wedge c')$ is a disjoint decomposition showing that $e \oplus g$ is defined, and $(b, c, b' \wedge c')$ is a disjoint decomposition showing that $f \oplus g$ is defined. As $e \oplus f$ is equal to $(a \vee b, a' \wedge b')$, to have $(e \oplus f) \oplus g$ defined, we would need a disjoint ternary decomposition whose first component was $a \vee b$, whose second component was c, and whose second and third components joined to $a' \wedge b$. But $a \vee b \vee c$ equals 1, and disjointness implies that $(a \vee b) \vee c$ meets with the third component to 0. Therefore the third component must be zero, a contradiction. $\square$

The construction described in this note can be modified. If one considers a category $\mathcal{C}$ with finite products where every ternary product diagram (e, f, g), disjoint or otherwise, produces a pushout $(e \times g, f \times g, \pi_2, \pi_2)$, then the collection of all binary decompositions, disjoint or otherwise, of an object in such a category produces an orthoalgebra. This construction applies to categories built from familiar mathematical structures such as non-empty sets, groups or topological spaces, but not a category built from a modular lattice. It is not known whether this construction always yields an orthomodular poset.

It may be possible to find some modification of the construction described in this note that generalizes not only the existing constructions of orthomodular posets from mathematical structures and modular lattices, but also the construction of orthomodular structures from symmetric lattices and relation algebras. Perhaps one could even find such a modification that always yielded orthomodular posets. This would seem to be a worthwhile direction of study.

REFERENCES

Abramsky, S. and Coecke, B. (2004). A categorical semantics of quantum protocols, *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science: LICS 2004*, IEEE Computer Society, pp. 415–425.

Birkhoff, G. and von Neumann, J. (1936). The logic of quantum mechanics. *Ann. Math.* **37**, 823–843.
Foulis, D. J., Greechie, R. J. and Ruttimann, G. T. (1992). Filters and supports on orthoalgebras. *Internat. J. Theoret. Phys.* **31**, 789–807.
Harding, J. (1996). Decompositions in quantum logic. *Trans. Amer. Math. Soc.* **348**(5), 1839–1862.
Harding, J. (1999). Axioms of an experimental system. *Internat. J. Theoret. Phys.* **38**(6), 1643–1675.
Isham, C. J. and Butterfield, J. (2000). Some possible roles for topos theory in quantum theory and quantum gravity, *Found. Phys.* **30**(10), 1707–1735.
Kalmbach, G. (1983). *Orthomodular Lattices*, Academic Press.
Mushtari, D. Kh. (1989). Projection logics in Banach spaces. *Soviet Math. (Iz. VUZ)* **33**, 59–70.
Pták, P. and Pulmannová, S. (1991). Orthomodular Structures as Quantum Logics. *Fundamental Theories of Physics*, **44**, Kluwer, Dordrecht.
Wilce, A. (2000). Test Spaces and Orthoalgebras, Current Research in Operational Quantum Logic, *Fund. Theories Phys.* **111**, Kluwer, Dordrecht 81–114.